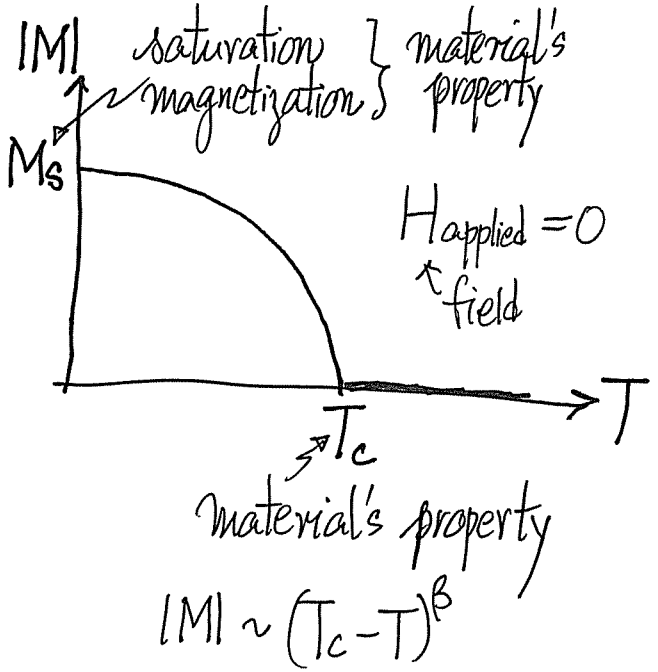
# X. Critical Phenomena: Ferromagnetism

## A. Key Features in Ferromagnetism

### Spontaneous Magnetization (自發)

- No external applied magnetic field

$$\begin{cases} M \neq 0 \text{ for } T < T_c \text{ (Curie temperature)} \\ \quad \text{"spontaneous" (∵ no applied field to magnetize sample)} \\ M = 0 \text{ for } T > T_c \end{cases}$$




Critical point:
$(T = T_c, H_{applied} = 0)$

| Material | $M_s$ ($10^6$ A/m) | $T_c$ (K) |
|---|---|---|
| Iron | 1.75 | 1043 |
| Cobalt | 1.45 | 1404 |
| Nickel | 0.512 | 631 |
| Gadolinium | 2.00 | 289 |
| Terbium | 1.44 | 230 |
| Dysprosium | 2.01 | 85 |
| Holmium | 2.55 | 20 |

*Sources: American Institute of Physics Handbook* (D. W. Gray, Ed.) (New York: McGraw-Hill, 1963).

- "Universal behavior" when properties of different Ferromagnetic (FM) materials are viewed in <u>reduced quantities</u>

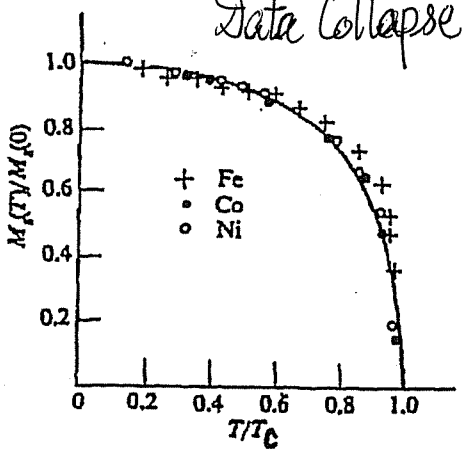
FM material A: Measure $M(T)$ $\Rightarrow M_S^{(A)}, T_c^{(A)}$

FM material B: Measure $M(T)$ $\Rightarrow M_S^{(B)}, T_c^{(B)}$

Look at $\underbrace{\frac{M(T)}{M_S} \text{ Vs } \frac{T}{T_c}}$

Almost the same curve for different materials

Data Collapse




- FOR $T \lesssim T_c$, $M \sim (T_c - T)^{\beta}$ and <u>same</u> $\beta$ for many different materials [c.f. vapor-liquid case]+
  - M is the <u>order parameter</u> of FM transition
  - M changes <u>continuously</u> at $T_c \Rightarrow$ "Continuous phase transition" (transitions that are <u>NOT</u> first order)

$T < T_c$, $M \neq 0$ Ferromagnetic phase (more ordered)

$T > T_c$, $M = 0$ Paramagnetic phase (disordered)

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+ Recall the law of corresponding state. Here is another example.

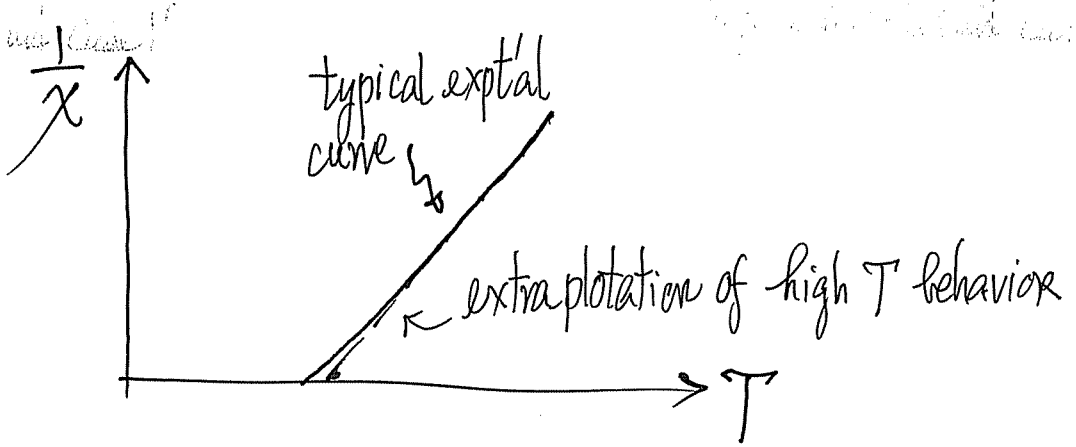
## Paramagnetic behavior at $T > T_c$

$T > T_c$, $H_{applied} = 0$, $M = 0$ (no spontaneous magnetization)

With an applied field, $M \neq 0$ when $H_{applied} \neq 0$

Response is described by $M = \chi H_{applied}$

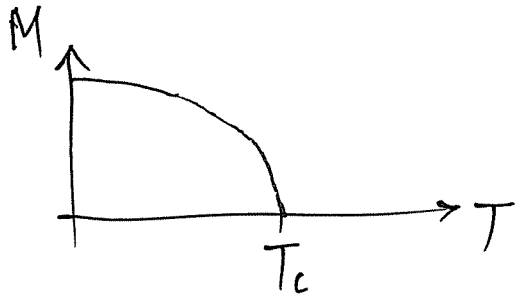
$\chi$ follows $\chi = \dfrac{C}{T - T_c}$ similar to Curie's law [recall $\chi \sim \frac{1}{T}$ for paramagnetic behavior]




It says, $\chi$ gets bigger and bigger as $T \to T_c$ from above.

As $T \to T_c$, $\chi$ diverges. OK! $M = \chi \underbrace{H_{applied}}_{=0}$ ($M \neq 0$) $\Rightarrow$ $\chi$ diverges when sample becomes FM!

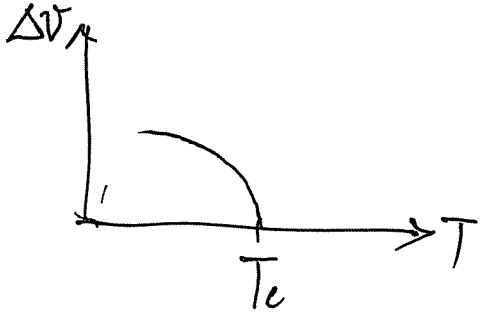
## Ferromagnetic-Paramagnetic Transition




Ordered $\leftrightarrow$ Disordered

$M \sim (T_c - T)^{\beta}$

## Liquid-vapor Transition




Ordered - Disordered

$\Delta v \sim (T_c - T)^{\beta'}$

It turns out that very different physical scenarios may carry the same value of critical exponent!

**This is what physicists meant by universal behavior.**

[Magnetic system and liquid-vapor system behave the same way near the critical point] (See Appendix A for Percolation Problem)

## B. Hints from theory of paramagnetism

Think like a physicist!

atoms → tiny magnets (magnetic moments)

$M \neq 0 \Rightarrow$ moments have tendency to align (even $H_{applied} = 0$)

Paramagnetism: No magnetic moment - magnetic moment interaction

& $\vec{\mu}$ only responds to $\vec{H}_{applied}$

Ferromagnetism: $\vec{\mu}$'s aligned! $\vec{\mu}$ must "feel" some local or internal magnetic field!

How come?

interact interact

Interaction between neighboring magnetic moments

a moment wants neighboring moments to point to its direction (ferromagnetic interaction)

Interaction: tend to align

kT : tend to randomize

low T or strong interaction $\Rightarrow$ alignment $\Rightarrow M \neq 0$

## How strong is the interaction?

Data: $T_c$ (iron) = 1043 K ~ $10^3$ K
$T_c$ (nickel) = 631 K

Meaning: $kT_c$ is not sufficient to randomize alignment

2 to 3 times of room temperature

$kT_c \sim$ 0.05 to 0.1 eV

strong interaction! [Much stronger than EM theory predicts]

The points are:

- Data indicate strong moment-moment interaction
  ⇒ alignment gains much energy
- Interaction has quantum origin
  ⇒ exchange interaction (due to anti-symmetric electron-wavefunction)
  electrons are fermions (a quantum idea)

# 1-page review on Paramagnetism

$J=\frac{1}{2}$ $(S=\frac{1}{2})$ — $+\mu_B B$ (anti-align with applied B)
— $-\mu_B B$ (align with applied B)
(two-level system)

$$Z = z^N, \quad z = e^{\beta\mu_B B} + e^{-\beta\mu_B B} = 2\cosh\left(\frac{\mu_B B}{kT}\right)$$

$$\langle \mu_z \rangle = \mu_B \tanh\left(\frac{\mu_B B}{kT}\right)$$

$\langle \mu_z \rangle$: one moment

Whole system: $N\langle \mu_z \rangle = N\mu_B \tanh\left(\frac{\mu_B B}{kT}\right)$

Per unit volume $M = \frac{N}{V}\langle \mu_z \rangle = \frac{N}{V}\mu_B \tanh\left(\frac{\mu_B B}{kT}\right) = M_S \tanh\left(\frac{\mu_B B}{kT}\right)$

$M_S$: saturation magnetization

$B$: B-field acting on each independent (non-interacting) magnetic moment

$\tanh\left(\frac{\mu_B B}{kT}\right)$: same as $B_J(x)$ with $J=\frac{1}{2}$

The underlying Hamiltonian is:

$$H_{para} = \sum_i -\vec{\mu}_i \cdot \vec{B}_{applied}$$

($i$: cover all magnetic moments)

with $\mu_{iz} = \begin{cases} +\mu_B & \text{(aligned with } \vec{B}_{applied}\text{)} \\ -\mu_B & \text{(anti-aligned with } \vec{B}_{applied}\text{)} \end{cases}$

## More on Paramagnetic Hamiltonian

The last equation of $H_{para}$ may as well be written as:

$$H_{para} = -(\mu_B B_{applied}) \sum_i \sigma_i \quad \text{with } \sigma_i = \begin{cases} +1 & \text{(align with } B_{applied}) \\ -1 & \text{(anti-align with } B_{applied}) \end{cases}$$

(over all magnetic moments)

$$= -(\mu_B B_{applied}) \sum_i S_i$$

(over all magnetic moments)

$(\mu_B B_{applied})$: an energy indicating how strong $B_{applied}$ is

both forms are used in textbooks and journal articles

Sometimes (e.g. in Ising model literature), this term is written as

$$H_{para} = -H \sum_i S_i \quad \text{OR} \quad -H \sum_i \sigma_i \quad \text{OR} \quad -B \sum_i \sigma_i \quad \text{OR} \quad -B \sum_i S_i$$

$H$: an energy representing $\mu_B B_{applied}$

$B$: an energy representing $\mu_B B_{applied}$

[unfortunate symbol, as H means many other things!]
(but must get used to it!)

## C. Modelling : An art and a science

- Good model : { include the essential physics in simplest form
  fewer symbols / simplified notations
  non-trivial results (eg. ferro to para transition)

Motivating a model :

- Essential physics - $\boxed{\vec{\mu}_i \text{ and } \vec{\mu}_j \text{ interaction } \underline{\text{vs}} \; kT}$

FM interaction : $\vec{\mu}_i$ and $\vec{\mu}_j$ align ($\mu_{iz}$ and $\mu_{jz}$ align)
$\Rightarrow$ lower energy
$\vec{\mu}_i$ and $\vec{\mu}_j$ anti-align ($\mu_{iz}$ and $\mu_{jz}$ anti-align)
$\Rightarrow$ higher energy

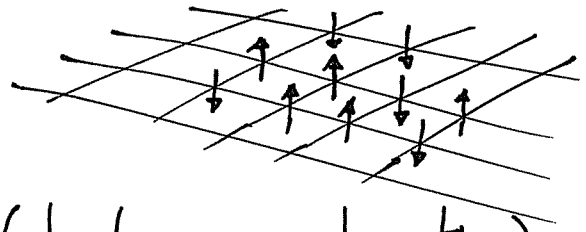
In general, $\vec{\mu}_i$ ($\vec{\mu}_j$) could have many $\hat{z}$-components (J quantum number).
Why bother! Grasp the essential physics { $+\mu_B$ / $-\mu_B$ } OR $J = \frac{1}{2}$ OR $S = \frac{1}{2}$ only

Simplest model: $\mu_z$ takes on only $\begin{cases} \mu_B(\text{"up"}) \\ -\mu_B(\text{"down"}) \end{cases}$ OR $\left\{ J_z = \pm 1/2 \right\}$ OR $\left\{ S_z = \pm 1/2 \right\}$

[Including more components will only add details]

Nearest-neighbor interaction only

- Interact only when two magnetic moments are nearest neighbors (short-range interaction)



2D square lattice

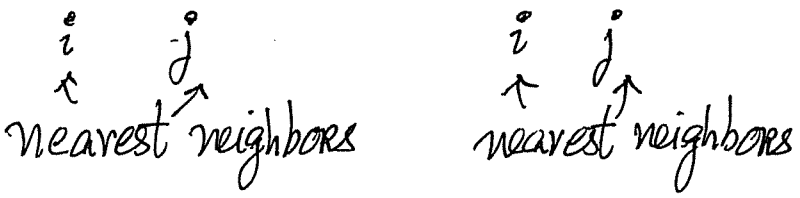
[Including longer-range interaction will only add details]

Big physics - short-range interaction can lead to long-range order

$\hat{z}$-component aligned: ↑ ↑ and ↓ ↓ $\Rightarrow E_{lower}$

$\hat{z}$-component anti-aligned: ↑ ↓ and ↓ ↑ $\Rightarrow E_{higher}$

differ by $E_{higher} - E_{lower} = "2J"$



In physical units, the <u>interaction energy</u> is of the form

$$-J_{ij}\,\underbrace{\vec{\mu}_i \cdot \vec{\mu}_j}_{\text{"}\pm\mu_B^2\text{"}\ \text{for}\ \begin{cases}\text{aligned}\\ \text{anti-aligned}\end{cases}}$$

$J_{ij}$: to make sure the expression is an energy

<u>Simplify notations</u>

- $\uparrow\ \uparrow$ as $+1\ +1$ or $+\ +$ | $\downarrow\ \downarrow$ as $-1\ -1$ or $-\ -$
  up up up up up up | down down down down down down
- $\uparrow\ \downarrow$ as $+1\ -1$ or $+\ -$ | $\downarrow\ \uparrow$ as $-1\ +1$ or $-\ +$
  up down up down up down | down up down up down up

<u>Recall</u>: Pauli matrices (spin-half)

$$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \Rightarrow \text{eigenvalues} = \underset{\uparrow}{+1}, \underset{\downarrow}{-1}$$

$\uparrow$ up, $\downarrow$ down

Write interaction energy as

$$-J_{ij}\, S_i S_j$$

$S_i$ $S_j$

with $\begin{cases} S_i \text{ taking on } +1 \text{ or } -1 \\ S_j \text{ taking on } +1 \text{ or } -1 \end{cases}$

aligned: $(+1,+1)$ and $(-1,-1)$ give $S_i S_j = 1$ and energy $-J_{ij}$

anti-aligned: $(+1,-1)$ and $(-1,+1)$ give $S_i S_j = -1$ and energy $+J_{ij}$

$\Big\}$ differ by $2J_{ij}$

- $J_{ij}$ is an energy characterizing the strength of interaction
- $J_{ij} > 0$, $S_i$ and $S_j$ tend to align $\Rightarrow$ FM interaction

[$J_{ij} < 0$, $S_i$ and $S_j$ tend to anti-align $\Rightarrow$ Anti-ferromagnetic interaction]

Assuming $J_{ij} = J$ (same) for all nearest-neighbors $(ij)$, the interaction energy (Hamiltonian) is

$$E(\{S_i\}) = -J \sum_{\langle ij \rangle} S_i S_j$$

$\sum_{\langle ij \rangle}$: sum over all distinct nearest-neighbor pairs

Ising model with no external applied magnetic field

In general, there may also be an applied (external) $\vec{B}_{applied}$.

An additional energy due to $(-\vec{\mu}_i \cdot \vec{B}_{applied})$ and sum up all $i$.

$\vec{B}_{applied} = B_{applied}\,\hat{z}$ (can always call the direction $\hat{z}$)

$$\text{Zeeman energy} = -\underbrace{\mu_B B_{applied}}_{\text{energy scale}} \overbrace{\sum_i S_i}^{\text{add up all magnetic moments}}$$

- contributing $-\mu_B B_{applied}$ (aligned)
- contributing $+\mu_B B_{applied}$ (anti-aligned)

$(S_i = +1 \text{ or } -1)$

Simplify notations

$$\text{Zeeman energy} = E_B(\{S_i\}) = -B \sum_i S_i$$

$B$: an energy ($\mu_B B_{applied}$) characterizing how strong the applied field is.

# D. Ising Model

$$E(\{S_i\}) = -J \sum_{\langle ij \rangle} S_i S_j - B \sum_i S_i \qquad (1)$$

- $-J \sum_{\langle ij \rangle} S_i S_j$: interaction between magnetic moments (or people called them spins)
- $-B \sum_i S_i$: energy representating external applied field

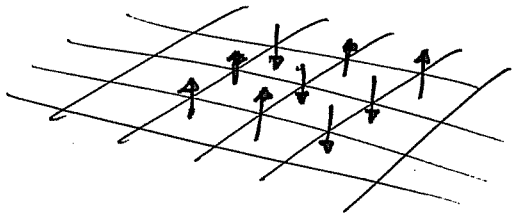
- this competes with $kT$
- $S_i$ takes on $+1$ or $-1$
- When $B=0$, can spins align even at finite temperature?
- Can study Ising model on 1D chain (exactly solvable), 2D lattices (some exactly solvable), 3D lattices (no exact solution), 4D lattices, ...
- Lenz (1920) constructed the model for Ising (1925) to study in his thesis

Reminder:

- $J = 0$, $E(\{S_i\}) = -B\sum_{i=1}^{N} S_i$ ($N$ spins or $N$ dipole moments) is the paramagnetic problem studied earlier
- $\langle S_i \rangle$ is an average number between $-1$ to $+1$
- Previously in paramagnetism, $\langle \mu_z \rangle = \mu_B \langle S_i \rangle$
- Now, with $J \neq 0$ in the Ising Model, we also want to find $\langle S_i \rangle$, which is proportional to the Magnetization.

## E. Ising Model: What can be done, formally?

$$E(\{S_i\}) = -J\sum_{\langle ij\rangle} S_i S_j - B\sum_i S_i \qquad (1)$$



Consider: N moments (spins) on a 2D square lattice

In Stat. Mech., want to evaluate $Z = \sum_{\text{all } N\text{-spin states } \{S_i\}} e^{-\frac{E(\{S_i\})}{kT}}$

- What to sum over?
  - $2^N$ strings of the form $\{S_1, S_2, \cdots, S_N\}$ with $S_i = \begin{cases} +1 \\ -1 \end{cases}$
- What goes into $e^{-\frac{E(\{S_i\})}{kT}}$?
  - For every string $\{S_i\}$, calculate $E(\{S_i\})$ from Eq. (1) and evaluate one term in $Z$
  - Repeat for each of $2^N$ strings ($2^N$ is a huge number for, say, $N = 100^2$)
  - Done! In principle.

## Can this be done?

- Analytically?
  - 1D chain: Yes (see Problem Set 3)
  - 2D square lattice: Yes (but not so easy)
  - 3D simple cubic or other lattice: No!
- Numerically?
  - Any dimension? (Write a program to evaluate $Z(T,N,B)$ exactly?) [huge # of $\{S_i\}$'s]
  - Any dimension? (An algorithm to carry out the importance sampling implied by the canonical ensemble) [Monte-Carlo simulation]
- Approximately?
  - Mean field theories
  - $1+\varepsilon$ ($\varepsilon \ll 1$) dimension; $4-\varepsilon$ ($\varepsilon \ll 1$) dimension
  - Renormalization methods